ESSENTIALLY NILPOTENT RINGS

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ABSTRACT

In this note we introduce a class of nil rings (called essentially nilpotent) which properly contains the class of nilpotent rings. A nil ring is said to be essentially nilpotent if it contains an essential right ideal which is nilpotent. Various properties of essentially nilpotent rings are investigated. A nil ring is essentially nilpotent if and only if it contains an essential right ideal which is left T-nilpotent.

1. Introduction

In this note we explore the idea of "approximating" nil rings by nilpotent rings. We are concerned with those nil rings which have a good approximation in the sense that they contain an essential right ideal L which is nilpotent, that is $L^n = (0)$ for some n and L has nonzero intersection with each nonzero right ideal of the ring. We extend this idea to cover right and left ideals of a ring. We say that a nil right (left) ideal of a ring is essentially right (left) nilpotent if it contains an essential right (left) ideal which is nilpotent. There are nil ideals which are essentially right but not left nilpotent; also, there are nil ideals which are neither essentially right nor left nilpotent. This paper investigates the properties of essentially right nilpotent ideals. Henceforth, *essentially nilpotent* will mean *essentially right nilpotent.* All nil right ideals which are not essentially nilpotent have certain properties in common: (Theorem 3.3) Let K denote a right ideal which is contained in the prime radical of a ring. If K is not essentially nilpotent then there does exist a sequence $\{x_i\}$ of K such that $x_1 \cdots x_n \neq 0$ for all n and the infinite sum $\Sigma x_i R$ is direct. Furthermore K does not satisfy the maximum condition on right annihilators. As a corollary we prove that a nil right ideal is essentially nilpotent if and only if it contains an essential right ideal which is left T-nilpotent. Recall that

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a right ideal L is *left T-nilpotent* if for each sequence $x_1, x_2, \cdots$ in L there does exist some n such that $x_1 \cdots x_n = 0$.

Throughout this paper R will always denote a ring. For a nonempty subset S in R let $r(S) = \{x \in R: sx = 0$ for all s in $S\}$. We call $r(S)$ the *right annihilator of* S. We similarly define the *left annihilator* $l(S)$ *of* S. For an element b in R we write $r(b)$ instead of $r(\{b\})$.

NOTATION. For $b \in R$ the symbol bR denotes the *principal right ideal generated* by the element b. We point out the set $\{bx: x \in R\}$ need not equal bR. The statement A is *essential in* B means that A and B are right ideals in R and $A \subseteq B$ and for each $b \in B - (0)$ there is $b' \in bR$ such that $b' \in A - (0)$. We will use the fact that if A is essential in B and A' is essential in A then A' is essential in B.

2. Definition and Examples

DEFINITION. A right ideal K in R is said to be essentially nilpotent if K contains a nilpotent right ideal which is essential in K. The following remarks are clear.

REMARK 1. If a right ideal is contained in some essentially nilpotent ideal then it is essentially nilpotent. Also if L is essentially nilpotent and L is essential in K then K is essentially nilpotent, where L and K are right ideals of R.

REMARK 2. A finite sum of essentially nilpotent ideals is essentially nilpotent. However an arbitrary sum need not be, see Example I of this paper.

Suppose a right ideal L is nilpotent. Then there is a positive integer n such that $a_1 \cdots a_n = 0$ for all $a_1, \cdots, a_n$ in L and $b_1 \cdots b_{n-1} \neq 0$ for some appropriate $b_1, \cdots, b_{n-1}$ in L. This unique n is called the index of the right nilpotent ideal L.

PROPOSITION 2.1. *A right ideal K in R is essentially nilpotent if and only if K contains a right ideal L in R such that L is essential in K and L is nilpotent with index two.*

PROOF. Suppose A is a nilpotent right ideal and A is essential in K. Then $L = l(A) \cap A$ is essential in A because A is nilpotent. Hence L is essential in K and $LL = (0)$. The other implication is clear.

PROPOSITION 2.2. *Let N denote a nil ring and $Z(N)$ its right singular ideal, that is, the set of elements whose right annihilators are essential in N. Then N is essentially nilpotent if and only if $Z(N)$ is essential in N and $Z(N)$ is essentially nilpotent.*

PROOF. Suppose that N is essentially nilpotent and the sum $Z(N)+H$ is direct where H is a nonzero right ideal of N. Let L be a nilpotent right ideal of N which is essential in N. We may assume that $LL=(0)$ by Proposition 2.1. Let $x \in H \cap L-(0)$. Since $x \in N - Z(N)$ there does exist a non zero right ideal K such that $K \cap r(x)=(0)$. Recall that L is essential so for some $y \in K$ we have $y \in L-0$; however, $0 \neq xy \in LL=(0)$, a contradiction. Therefore, $H=(0)$ and the proposition follows.

EXAMPLE I. This example is due to Sasiada [4]. Let R denote the ring generated by $x_1, x_2, \cdots$ with the relations $x_i x_j = 0$ for $i \geqq j$. The ring R is nil and the left singular ideal is zero; the right singular ideal Rx_1R is nilpotent and is an essential right ideal in R. So R is essentially right but not left nilpotent. Also $R = \Sigma Rx_iR$, where Rx_iR is the ideal generated by x_i and $(Rx_iR)(Rx_iR)=(0)$. Thus, R is a sum of essentially left nilpotent ideals but is not essentially left nilpotent, see Remark 2.

EXAMPLE II. Let B denote a nil semiprime ring—for example the famous ring due to R. Baer [1] or [2, p. 46]. Thus, B is not essentially right nilpotent and is not essentially left nilpotent. Let A be the ring generated by x and all elements of B with the relations $xx=0$ and $xb=bx$ for all $b \in B$. The ring A is essentially nilpotent since the right ideal generated by x is $Z(A)$, the right singular ideal, and $Z(A)$ is both essential and nilpotent. Also the prime radical of $A \neq A$. The Sasiada ring R of Example I is not left essentially nilpotent but the prime radical of R is R. We conclude that there is no containment relationship between the class of essentially nilpotent nil rings and the class of nil rings with the property that each nil ring is its own prime radical.

3. The Main Theorem

We define the prime radical of a ring as the intersection of the prime ideals in R. Let P denote the prime radical of R. Also P contains all nilpotent right ideals in R. Thus a right ideal L is essentially nilpotent if and only if the intersection $L \cap P$ is essentially nilpotent and $L \cap P$ is essential in L. Recently, J. Lambek [7, p. 55] has described the prime radical as the set of strongly nilpotent elements. Recall that an element x is strongly nilpotent if each sequence $x_0, x_1, x_2, \cdots$ is ultimately zero where $x_0 = x$ and x_{n+1} is in $\{x_n y x_n \colon y \in R\}$. Hence if x is in the prime radical there is some $z \in xR-(0)$ such that $zyz=0$ for all $y \in R$. We use this fact below.

PROPOSITION 3.1. *Let L be a right ideal contained in the prime radical of R. Then L contains a direct sum $B = \sum x_i R$ where $i \in I$ and each $x_i y x_i = 0$ for all $y \in R$. Furthermore B is essential in L.*

PROOF. Let C denote the collection of independent sets of right ideals of the form $\{b_i R : i \in I'\}$ where for $i \in I'$ we have $b_i y b_i = 0$ for all $y \in R$. By Zorn's Lemma we select a maximal element $\{b_i R : i \in I \subseteq I'\}$ in C: let $B = \sum b_i R$ where $i \in I$. If B is not essential in L then for some $x \in L - (0)$ the sum $B + xR$ is direct. Since x is strongly nilpotent there is $z \in xR - (0)$ such that $zyz = 0$ for all $y \in R$. The sum $B + zR$ is direct and thus $\{b_i R : i \in I\} \cup \{zR\}$ is in C, a contradiction. Therefore B is essential in L and this completes the proof.

COROLLARY 3.2. *The prime radical is essentially nilpotent in a commutative ring.*

PROOF. By the above proposition there is a direct sum $B = \sum x_i R$ where $i \in I$ and $(x_i R)^3 = (0)$. Furthermore, B is essential in the prime radical. For $i \neq j$ in I we have $(b_i R)(b_j R) \subseteq b_i R \cap b_j R = (0)$ since R is commutative and the sum $b_i R + b_j R$ is direct. Recall $(x_i R)^3 = (0)$ for all $i \in I$. Hence, $B^2 = (0)$ and B is nilpotent.

THEOREM 3.3. *Let L denote a right ideal contained in the prime radical of R. If L is not essentially nilpotent then there does exist a sequence $x_1, x_2, \cdots$ in L such that $x_1 \cdots x_n \neq 0$ for all n and the infinite sum $x_1 R + x_2 R + \cdots$ is direct. Furthermore, $r(S_p) \subsetneqq r(S_{p+1})$ for all $p \geqq 1$ where $S_p = \{x_1 \cdots x_p,\ x_1 \cdots x_p x_{p+1}, \cdots\}$ and $S_{p+1} + S_p = \{x_1 \cdots x_p\}$.*

PROOF. From Proposition 3.1 there does exist a direct sum $B = \sum b_i R$ for $i \in I$ such that $b_i y b_i = 0$ for all $y \in R$ and B is essential in L. Let $K = l(B) \cap B$ where $l(B)$ is the left annihilator of B and thus $KK = (0)$. If K were essential in B which is essential in L then L would be essentially nilpotent, a contradiction. Therefore, for some $x \in B - (0)$ the sum $xR + K$ is direct. Assume that there does exist a finite sequence $x_1, x_2, \cdots, x_n$ of B such that $x x_1 \cdots x_n \neq 0$ and each $x_i = b_{k_i}$ for some $k_i \in I$ and thus $x_i R$ is a summand of B. If $x x_1 \cdots x_n b_i = 0$ for all $i \in I$ then $x x_1 \cdots x_n \in K$, a contradiction. Therefore, $x x_1 \cdots x_n b_j \neq 0$ for some $j \in I$ and let $x_{n+1} = b_j$. We conclude that there does exist a sequence $\{x_i\}$ in B such that $x_1 \cdots x_n \neq 0$ and each $x_n R$ is a summand of B; also $x_n y x_n = 0$ for all $y \in R$. Immediately, $i < j$ implies $x_i \neq x_j$ otherwise $x_i \cdots x_j \in \{x_i y x_i : y \in R\} = \{0\}$, a contradiction. The sum $\sum x_i R$ is direct and infinite. Also, $x_1 \cdots x_n x_h = 0$ for $1 \leqq h \leqq n$. Therefore $x_{p+1} \in r(S_{p+1}) - r(S_p)$ where S_p is defined in the hypothesis.

COROLLARY 3.4. *If R satisfies the maximum condition on right annihilators then the prime radical is essentially nilpotent.*

PROOF. This is clear.

COROLLARY 3.5. *The prime radical is essentially nilpotent in a right finite dimensional ring.*

PROOF. This is clear.

COROLLARY 3.6. *A right ideal is essentially nilpotent if and only if it contains an essential right ideal which is left T-nilpotent*

PROOF. Let N denote a right ideal. If N is essentially nilpotent then the result is clear. For the other implication suppose that N contains an essential right ideal L which is left T-nilpotent. Since L is left T-nilpotent each element of L is strongly nilpotent. It follows from Theorem 3.3 that L is essentially nilpotent and thus N is also by Remark 1.

We quote from [5, p. 255]: "The question as to whether or not the (Jacobson) radical of every finitely generated algebra is nil remains open." To the author's knowledge the question as to whether or not each matrix ring over a nil ring remains a nil ring is an open question. The property of a ring being essentially nilpotent is carried over to matrix rings.

PROPOSITION 3.7. *Let N be a ring. Then N is essentially nilpotent if and only if each matrix ring over N is essentially nilpotent.*

PROOF. Suppose N is essentially nilpotent. If L is essential in N then the ideal generated by elements of the form $\sum e_{ij}x_{ij}$, where $x_{ij} \in L$ and e_{ij}'s are the matrix units, is essential in the matrix ring over N. The implication now follows. If K is essential in the matrix ring over N then $K \cap \sum e_{1j}N$ is essential in $\sum e_{1j}N$. The other implication is now clear.

Let P denote the prime radical of a ring R. Recently it has been shown that P is nilpotent if and only if the annihilating conditions on P hold: for some n, $r(P^n) \cap P = r(P^{n+1}) \cap P$ and for each sequence $\{x_i\}$ of P there does exist some m such that $l(Rx_m \cdots x_1R) \cap P = l(Rx_{m+t} \cdots x_1R) \cap P$ for all $t \geqq 1$ where $Rx_m \cdots x_1R$ denotes the ideal generated by $x_m \cdots x_1$ [8].

It is known that P is nilpotent if and only if an arbitrary sum of nilpotent ideals is nilpotent. A nilpotent ideal is essentially nilpotent. Thus if an arbitrary sum of essentially nilpotent ideals in P is nilpotent then P is nilpotent. Conversely if P is nilpotent then any arbitrary sum of ideals in P is nilpotent and thus essentially nilpotent. The author does not know of any significant condition (which would involve essentially nilpotent ideals) for a nil ring to be nilpotent.

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